A singular Hamilton-Jacobi equation modeling the tail problem

Sepideh Mirrahimi * Guy Barles † Benoît Perthame *‡ Panagiotis E. Souganidis §¶

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Abstract

We study the long-time/long-range behavior of reaction diffusion equations with negative square root-type reaction terms. In particular we investigate the exponential behavior of the solutions after space and time are scaled in a hyperbolic way by a small parameter. This leads to a new type of quasi-variational inequality for a Hamilton-Jacobi equation. The novelty is that the obstacle, which defines the open set where the solutions of the reaction diffusion equation do not vanish in the limit, depends on the solution itself. Counter-examples show a nontrivial lack of uniqueness for the variational inequality depending on the conditions imposed on the free boundary of this open set. Both Dirichlet and state constraints boundary conditions play a role. When the competition term does not change sign, we can identify the limit while, in general, we only obtain lower and upper bounds.

Although models of this type are rather old and extinction phenomena are as important as blow-up, our motivation comes from the so-called "tail problem" in population biology. One way to avoid meaningless exponential tails is to impose a singular mortality rate below a given survival threshold. Our study shows that the precise form of this singular mortality term is asymptotically irrelevant and that, in the survival zone, the population profile is impacted by the survival threshold (except in the very particular case when the competition term is nonpositive).

Key-words: Reaction-diffusion equations, Asymptotic analysis, Hamilton-Jacobi equation, Survival threshold, Population biology, quasi-variational inequality, Free boundary.
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1 Introduction

We study the asymptotic behavior, as $\varepsilon \to 0$, of the solutions to reaction-diffusion equations (with singular reaction term) of the form

$$\begin{cases} n_{\varepsilon,t} - \varepsilon \Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R - \frac{1}{\varepsilon}(\beta_\varepsilon n_\varepsilon)^{1/2} & \text{in } \mathbb{R}^d \times (0, +\infty), \\ n_\varepsilon = e^{u_\varepsilon^0/\varepsilon} \quad \text{on} \quad \mathbb{R}^d \times \{0\}, \end{cases} \tag{1}$$

*UPMC, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris. Email: mirrahimi@ann.jussieu.fr
†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Fédération Denis Poisson, Université François Rabelais, Parc de Grandmont, 37200 Tours, France. Email: barles@lmpt.univ-tours.fr
‡INRIA EPI BANG and Institut Universitaire de France. Email: benoit.perthame@upmc.fr
§The University of Chicago, Department of Mathematics, 5734 S. University Avenue, Chicago, IL 60637, USA. Email: souganidis@math.uchicago.edu
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where $u_\varepsilon^0 : \mathbb{R}^d \to \mathbb{R}$ is a given function, $R : \mathbb{R}^d \to \mathbb{R}$ represents a linear logistic growth/death rate and the survival threshold parameter β, which models a singular death term, is given, for some $u_m < 0$, by

$$\beta_\varepsilon = e^{u_m/\varepsilon}. \tag{2}$$

The positive parameter ε is introduced by a hyperbolic scaling $(x,t) \mapsto (x/\varepsilon, t/\varepsilon)$ with the aim to describe the long time and long range behavior of the unscaled problem (corresponding to $\varepsilon = 1$). The limiting behavior of scaled reaction-diffusion equations with KPP-type reaction has been studied extensively in, among other places, the theory of front propagation ([4, 19, 12]) using the so called WKB-(exponential) change of the unknown.

The novelty of the problem we are considering here is the presence of the negative square root term. To the best of our knowledge, the first study of such nonlinearity goes back to [11] where it is proved that local extinction occurs, i.e., the solution can vanish in a domain and stay positive in another region. For this reason β is thought to represent a survival threshold. That a solution of a parabolic problem can vanish locally is a surprising effect and as singular as the blow-up phenomena for supercritical reactions terms ([17]). In population biology such behavior prevents the so-called "tail problem" where very small (and thus meaningless) populations can generate artifacts ([14]). Although the mathematical analysis of the limit of (1) turns out to be a full subject in itself, our primary motivation comes from qualitative questions in population dynamics.

Indeed (1) is the simplest model for studying the effect of "cutting the tail" but many other problems are relevant in ecology. Along the same lines, in the context of front propagation, one may consider the modified Fisher–KPP equation

$$n_{\varepsilon,t} - \varepsilon \Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon (1 - n_\varepsilon) - \frac{1}{\varepsilon} (\beta_\varepsilon n_\varepsilon)^{1/2} \quad \text{in } \mathbb{R}^d \times (0, +\infty),$$

and ask the question whether the square root term changes fundamentally the study in [12] and [14] of the propagation of the invading/combustion fronts. In the context of speciation, an elementary model in adaptive evolution is the non-local reaction-diffusion equation

$$n_{\varepsilon,t} - \varepsilon \Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R(x, I_\varepsilon) - \frac{1}{\varepsilon} (\beta_\varepsilon n_\varepsilon)^{1/2} \quad \text{in } \mathbb{R}^d \times (0, +\infty) \quad \text{with} \quad I_\varepsilon(t) = \int \psi(x) n_\varepsilon(x,t) dx,$$

where n_ε is the population density of individuals with phenotypical trait x, R represents the net growth rate, ψ is the consumption rate of individuals and $I(t)$ is the total consumption of the resource at time t. The survival threshold was introduced in [14]. Finally ε may represent large time and small mutations as studied in [5, 6, 16]. It is known that under some assumptions the density concentrates as an evolving Dirac mass for the fittest trait. In biological terms this means that one or several dominant traits survive while others become extinct. Phenomena such as the discontinuous jumps of the fittest trait, non smooth branching and fast dynamics compared to stochastic simulations, motivated [14] to improve the model by including a survival threshold. Numerical results confirm that this modification gives dynamics comparable to stochastic models. It is interesting to investigate rigorously whether the dynamics of the Dirac concentration points are really changed by the survival threshold and to explain why its specific form ($n_\varepsilon^{1/2}$ versus n_ε^γ with $0 < \gamma < 1$) seems irrelevant.

A way to approach these questions for (1) is through the asymptotic analysis of n_ε. Since, as in the classical case, i.e., the Fisher-KPP equation without the square root term (see [12]), n_ε decays exponentially, the limit is better described using the Hopf-Cole transformation

$$u_\varepsilon = \varepsilon \ln n_\varepsilon, \tag{3}$$

which, for $u_\varepsilon^0 = \varepsilon \ln n_\varepsilon^0$, leads to the "viscous" Hamilton-Jacobi initial value problem

$$\begin{cases} u_{\varepsilon,t} - \varepsilon \Delta u_\varepsilon - |Du_\varepsilon|^2 = R - \exp\left(\dfrac{u_m - u_\varepsilon}{2\varepsilon}\right) \quad \text{in} \quad \mathbb{R}^d \times (0, +\infty), \\ u_\varepsilon = u_\varepsilon^0 \quad \text{in } \mathbb{R}^d \times \{0\}. \end{cases} \tag{4}$$

Throughout the paper we assume that there exist $C > 0$ and $u^0 \in \mathrm{C}^{0,1}(\mathbb{R}^d)$ such that

$$\|R\|_{\mathrm{C}^{0,1}} \leq C \quad \text{and} \quad \|u^0\|_{\mathrm{C}^{0,1}} \leq C, \tag{5}$$

$$u_\varepsilon^0 \in \mathrm{C}(\mathbb{R}^d), \quad u_\varepsilon^0 \leq C \quad \text{and} \quad u_\varepsilon^0 \underset{\varepsilon \to 0}{\longrightarrow} u^0 \quad \text{in} \quad \mathrm{C}(\mathbb{R}^d). \tag{6}$$

In the limit $\varepsilon \to 0$, it is easy to see, at least formally, that any local uniform limit of the family $(u_\varepsilon)_{\varepsilon>0}$ will satisfy, in the sense of the Crandall-Lions viscosity solutions ([10]), the Hamilton-Jacobi free boundary problem

$$\begin{cases} u_t = |Du|^2 + R \quad \text{in} \quad \Omega \subset \mathbb{R}^d \times (0, \infty), \\ u = -\infty \quad \text{in} \quad \overline{\Omega}^c \cap (\mathbb{R}^d \times (0, \infty)), \\ u \geq u_m \quad \text{in} \quad \overline{\Omega}, \\ u = u^0 \quad \text{in} \quad \overline{\Omega} \cap (\mathbb{R}^d \times \{0\}), \end{cases} \tag{7}$$

with the space-time open set Ω defined by

$$\Omega = \mathcal{I}nt \left\{(x,t) \in \mathbb{R}^d \times (0,\infty) : \lim_{\varepsilon \to 0} u_\varepsilon(x,t) > -\infty\right\}.$$

Notice that (7) is an obstacle problem with an obstacle depending on the solution itself. As a matter of fact the open set Ω plays an important role and, hence, the problem may be better stated in terms of the pair (u, Ω). The difficulty is that (7) has several viscosity solutions (see Appendix A for examples) depending on the sense the boundary conditions are achieved and the sign of R.

Next we discuss the two boundary conditions arising in (7). The first is the Dirichlet boundary condition in the third relation in (7). Its precise form is

$$\lim_{(x,t) \to (x_0,t_0) \in \partial\Omega} u(x,t) = u_m. \tag{8}$$

The second is the state constraint boundary condition (see [18]), which is natural in view of the second equality in (7). It states that

$$u \text{ is a supersolution in } \overline{\Omega} \text{ and a subsolution in } \Omega\ . \tag{9}$$

The basic questions we are considering in this paper are:

• What boundary condition should be satisfied by the limits of the family $(u_\varepsilon)_{\varepsilon>0}$ on $\partial\Omega$? Dirichlet or state constraint? The latter appears to play a fundamental role. To the best of our knowledge, there are no results available for state constraint problems with time varying and non smooth domains. Most of the technicalities in the paper stem from this difficulty.

• Does the limit $\varepsilon \to 0$ select a particular solution to (7), i.e., is there a natural selection? Is the limit of the family $(u_\varepsilon)_{\varepsilon>0}$ the maximal subsolution or minimal solution to (7)?

• Do the limits of the family $(u_\varepsilon)_{\varepsilon>0}$ depend on the specific form of the survival threshold, i.e., can we replace $(\beta_\varepsilon n_\varepsilon)^{1/2}$ by $(\beta_\varepsilon n_\varepsilon)^\gamma$ with $\gamma \in (0,1)$ without affecting the outcome?

An important ingredient of our analysis is the asymptotics, as $\varepsilon \to 0$, of the solution u^1_ε of

$$\begin{cases} u^1_{\varepsilon,t} = \varepsilon \Delta u^1_\varepsilon + |Du^1_\varepsilon|^2 + R \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\ u^1_\varepsilon = u^0_\varepsilon \quad \text{in } \mathbb{R}^d \times \{0\}, \end{cases} \tag{10}$$

which is obtained, after the Hopf-Cole transformation

$$u^1_\varepsilon = \varepsilon \ln n^1_\varepsilon, \tag{11}$$

from the simplified reaction diffusion equation

$$\begin{cases} n^1_{\varepsilon,t} - \varepsilon \Delta n^1_\varepsilon = \frac{n^1_\varepsilon}{\varepsilon} R \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\ n^1_\varepsilon = \exp(\varepsilon^{-1} u^0_\varepsilon) \quad \text{in } \mathbb{R}^d \times \{0\}. \end{cases} \tag{12}$$

In view of (5) and (6), it follows from [12] that, as $\varepsilon \to 0$, the sequence $(u^1_\varepsilon)_\varepsilon$ converge locally uniformly to $u^1 \in \mathrm{C}(\mathbb{R}^d \times (0,\infty))$, which is the unique viscosity solution of the eikonal -type equation

$$\begin{cases} u^1_t = |Du^1|^2 + R \quad \text{in } \mathbb{R}^d \times (0,+\infty), \\ u^1 = u^0 \quad \text{in } \mathbb{R}^d \times \{0\}. \end{cases} \tag{13}$$

The maximum principle yields $n_\varepsilon \le n^1_\varepsilon$, which in turn implies that $u_\varepsilon \le u^1_\varepsilon$ and, in the limit (this is made precise later), $u \le u^1$. It also follows from (4), at least formally, that, as $\varepsilon \to 0$,

$$u_\varepsilon \to -\infty \qquad \text{in } (\mathbb{R}^d \times (0,\infty)) \backslash \overline{\Omega^1},$$

where

$$\Omega^1 = \{(x,t) \mid u^1(x,t) > u_m\}. \tag{14}$$

It turns out that the case of nonpositive rate R is particularly illuminating and the above questions can be answered completely and positively using u^1 (see Section 2). The problem is, however, considerably more complicated when R takes positive values. In this case we introduce an iterative procedure that builds sequences of sub and supersolutions (Section 3). This construction gives the complete limit of u_ε when R is constant (Section 4). The limit is not the maximal subsolution of (7) and the Dirichlet condition is not enough to select it. In Section 5, we consider strictly positive spatially dependent R and provide a complete answer in terms of the iterative procedure. The relative roles of the Dirichlet and state constraint boundary conditions appear clearly in this case. In Section 6 we summarize our results. In the three part Appendix we present some examples of nonuniqueness as well as the proofs of few technical facts used earlier.

We conclude the introduction with the definition and the notation of the half-relaxed limits that we will be using throughout the paper. To this end, if $(w_\varepsilon)_{\varepsilon>0}$ is a family of bounded functions, the upper and lower limits, which are denoted by $\bar{w}$ and $\underline{w}$ respectively, are given by

$$\overline{w}(x) = \limsup_{\varepsilon \to 0, y \to x} w_\varepsilon(y) \quad \text{and} \quad \underline{w}(x) = \liminf_{\varepsilon \to 0, y \to x} w_\varepsilon(y). \tag{15}$$

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2 Nonpositive growth rate

Here we assume

$$R \leq 0 \quad \text{in} \quad \mathbb{R}^d, \tag{16}$$

and show that the behavior of the family $(u_\varepsilon)_\varepsilon$, in the limit $\varepsilon \to 0$, can be described completely in terms of solution u^1 of (13), which carries all the necessary information. More precisely, we can state the

Theorem 2.1. *Assume (5), (6) and (16). As $\varepsilon \to 0$, the family $(u_\varepsilon)_{\varepsilon>0}$ converges, locally uniformly in Ω^1 and in $\left(\mathbb{R}^d \times (0,\infty)\right) \setminus \overline{\Omega^1}$, to*

$$u(x,t) = \begin{cases} u^1(x,t) & \text{for} \quad (x,t) \in \Omega^1, \\ -\infty & \text{for} \quad (x,t) \in \left(\mathbb{R}^d \times (0,\infty)\right) \setminus \overline{\Omega^1}, \end{cases} \tag{17}$$

with u^1 and Ω^1 defined by (13) and (14) respectively. In particular, $u(x,t) \to u_m$ as $(x,t) \to \partial\Omega^1$.

Before we begin with the proof, we present and discuss below several remarks and observations which are important to explain the meaning of the results.

Firstly, by "uniform convergence" to $-\infty$, we mean $\limsup_{\varepsilon\to 0, y\to x, s\to t} u_\varepsilon(y,s) = -\infty$. Secondly, the u associated with the open set Ω^1 is the maximal solution to (7). Indeed any other solution $\widetilde{u}$, with the corresponding open set $\widetilde{\Omega}$, satisfies $\widetilde{u} \leq u^1$ and thus $\widetilde{\Omega} \subset \Omega^1$ and $\widetilde{u} \leq u$. It also satisfies the Dirichlet and state constraint boundary conditions. To verify the latter we notice, using the standard optimal control formula ([15, 13, 1]), that

$$u^1(x,t) = \sup_{\substack{(x(s),s)\in\mathbb{R}^d\times[0,\infty) \\ x(t)=x}} \left\{ \int_0^t \left(-\frac{|\dot{x}(s)|^2}{4} + R(x(s)) \right) ds + u_0(x(0)) : x \in C^1([0,t];\mathbb{R}^d) \right\}.$$

If $\widetilde{x}(\cdot)$ is an optimal trajectory, the dynamic programming principle implies that, for any $0 < \tau < t$,

$$u^1(x,t) = \int_\tau^t \left(-\frac{|\dot{\widetilde{x}}(s)|^2}{4} + R(\widetilde{x}(s)) \right) ds + u^1(\widetilde{x}(\tau),\tau).$$

Since R is nonpositive, u^1 is decreasing along the optimal trajectory. It follows that, if $u^1(x,t) > u_m$, then, for all $0 \leq \tau < t$, $u^1(\widetilde{x}(\tau),\tau) > u_m$.

Hence, for all $(x,t) \in \Omega^1$,

$$u(x,t) = \sup_{\substack{(x(s),s)\in\Omega^1 \\ x(t)=x}} \left\{ \int_0^t \left(-\frac{|\dot{x}(s)|^2}{4} + R(x(s)) \right) ds + u_0(x(0)) : x \in C^1([0,t];\mathbb{R}^d) \right\},$$

and, therefore, u verifies the state constraint condition.

Finally, the limit u does not depend on the details of the singular death term. In particular it is the same if we replace in (1) $n_\varepsilon \exp(\varepsilon^{-1}u_m)^{1/2}$ by $n_\varepsilon^\gamma \exp(\varepsilon^{-1}\gamma u_m)$ with $0 < \gamma < 1$. Hence, the value $\gamma = 1/2$ is irrelevant.

We continue with the

Proof of Theorem 2.1. As already discussed in the introduction, we know that $u_\varepsilon \le u^1_\varepsilon$ but we cannot obtain directly the other inequality in the limit $\varepsilon \to 0$. It is therefore necessary to introduce a pair of auxiliary functions v^A_ε and $v^{A,1}_\varepsilon$ which converge, as $\varepsilon \to 0$, in $\mathrm{C}(\mathbb{R}^d \times (0,\infty))$ to $\max(u^1, -A)$. Using this information for appropriate values of the parameter A, we then prove that, as $\varepsilon \to 0$, $u_\varepsilon \to u^1$ locally uniformly in the open set

$$\mathcal{A} = \{(x,t) :\ u^1(x,t) > u_m\}, \tag{18}$$

and $u_\varepsilon \to -\infty$ locally uniformly in the open set

$$\mathcal{B} = \{(x,t) :\ u^1(x,t) < u_m\}. \tag{19}$$

To this end, for any A such that

$$0 < A < -u_m, \tag{20}$$

we consider the functions v^A_ε and $v^{A,1}_\varepsilon$ given by

$$n_\varepsilon + \exp(\frac{-A}{\varepsilon}) = \exp(\frac{v^A_\varepsilon}{\varepsilon}) \quad \text{and} \quad n^1_\varepsilon + \exp(\frac{-A}{\varepsilon}) = \exp(\frac{v^{A,1}_\varepsilon}{\varepsilon}). \tag{21}$$

We have:

Proposition 2.2. *Assume (5), (6), (16) and (20). As $\varepsilon \to 0$, the families $(v^{A,1}_\varepsilon)_{\varepsilon>0}$ and $(v^A_\varepsilon)_{\varepsilon>0}$ converge in $\mathrm{C}(\mathbb{R}^d \times [0,\infty))$ to the unique solution $v^{A,1} = \max(u^1, -A)$ of*

$$\begin{cases} \min\big(v^{A,1} + A, v^{A,1}_t - |Dv^{A,1}|^2 - R\big) = 0 \quad \textit{in} \ \ \mathbb{R}^d \times (0,\infty), \\[2mm] v^{A,1} = \max(u^0, -A) \quad \textit{on} \ \ \mathbb{R}^d \times \{0\}. \end{cases} \tag{22}$$

We postpone the proof to the end of this section and next we prove the convergence of the family $(u_\varepsilon)_\varepsilon$ in the sets $\mathcal{A}$ and $\mathcal{B}$. We begin with the former.

Fix $(x_0,t_0) \in \mathcal{A}$. By the definition of $\mathcal{A}$ we have $u^1(x_0,t_0) > u_m$ and, hence, we can choose A such that $u^1(x_0,t_0) > -A > u_m$. Proposition 2.2 yields that, as $\varepsilon \to 0$ and uniformly in any neighborhood of (x_0,t_0),

$$v^A_\varepsilon \to v^{A,1} = \max(-A, u^1) = u^1.$$

Using the latter, the choice of A and the fact that

$$u_\varepsilon = v^A_\varepsilon + \varepsilon \ln\big(1 - \exp(\varepsilon^{-1}(-A - v^A_\varepsilon))\big),$$

we deduce that, as $\varepsilon \to 0$, $u_\varepsilon \to u^1$ uniformly in any neighborhood of (x_0,t_0).

Next we consider the limiting behavior in the set $\mathcal{B}$. To this end, observe that, using (3) and (11), we find $u_\varepsilon \le u^1_\varepsilon$ and, thus, passing to the limit in the viscosity sense, $\overline{u} \le u^1$ and

$$\overline{u} < u_m \quad \text{in} \quad \mathcal{B}.$$

Assume that, for some $(x_0,t_0) \in \mathcal{B}$, $\overline{u}(x_0,t_0) > -\infty$. Since $\overline{u}$ is upper semicontinuous (see [2]), there exists a family $(\phi_\alpha)_{\alpha>0}$ of smooth functions such that $\overline{u} - \phi_\alpha$ attains a strict local maximum at some (x_α,t_α) and, as $\alpha \to 0$,

$$(x_\alpha,t_\alpha) \to (x_0,t_0), \quad \overline{u}(x_\alpha,t_\alpha) \ge \overline{u}(x_0,t_0) \quad \text{and} \quad \overline{u}(x_\alpha,t_\alpha) \to \overline{u}(x_0,t_0).$$

It follows that there exists points $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$ such that $u_\varepsilon - \phi_\alpha$ attains a local maximum at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$, $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \to (x_\alpha, t_\alpha)$ as $\varepsilon \to 0$, and, in view of (4), at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$,

$$\phi_{\alpha,t} - \varepsilon\Delta\phi_\alpha - |D\phi_\alpha|^2 - R \le -\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon)).$$

Letting $\varepsilon \to 0$ we find that at (x_α, t_α)

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \le \limsup_{\varepsilon\to 0}[-\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})))].$$

The definition of $\overline{u}$ yields

$$\limsup_{\varepsilon\to 0} u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \le \overline{u}(x_\alpha, t_\alpha)$$

and, since, for α sufficiently small, $\overline{u}(x,t) < u_m$, we have

$$\overline{u}(x_\alpha, t_\alpha) < u_m \quad \text{and} \quad \limsup_{\varepsilon\to 0}[-\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})))] = -\infty$$

and, finally, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \le -\infty,$$

which is not possible because ϕ_α is a smooth function.

The claim about the uniform convergence on compact subsets is an immediate consequence of the upper semicontinuity of $\overline{u}$ and the previous argument. $\square$

We conclude the section with the proof of Proposition 2.2. Since it is long, before entering in the details, we briefly describe the main steps. We begin by establishing independent of ε bounds on the family $(v_\varepsilon^A)_\varepsilon$. Then we show that the half-relaxed limits $\overline{v}^\alpha$ and $\underline{v}^\alpha$ are respectively sub and supersolutions of (22). We conclude by identifying the limit.

Proof of Proposition 2.2. By the definition of v_ε^A, we have $v_\varepsilon^A > -A$ and, thus, the family $(v_\varepsilon^A)_\varepsilon$ is bounded from below.

To prove an upper bound we first notice that, on $\mathbb{R}^d \times \{0\}$,

$$v_\varepsilon^A = u_\varepsilon^0 + \varepsilon \ln(1 + e^{\frac{-A-u_\varepsilon^0}{\varepsilon}}) \quad \text{and} \quad v_\varepsilon^A = -A + \varepsilon \ln(1 + e^{\frac{A+u_\varepsilon^0}{\varepsilon}}), \tag{23}$$

hence,

$$v_\varepsilon^A \le \max(u_\varepsilon^0 + \varepsilon\ln(2), -A + \varepsilon\ln(2)) \quad \text{on} \quad \mathbb{R}^d \times \{0\},$$

and, finally, in view of (6),

$$v_\varepsilon^A \le C_A \quad \text{on} \quad \mathbb{R}^d \times \{0\},$$

for $C_A > 0$ such that $\max(-A, u_\varepsilon^0) \le C_A$.

Moreover, since $R \le 0$, we have

$$v_{\varepsilon,t}^A - \varepsilon\Delta v_\varepsilon^A - |Dv_\varepsilon^A|^2 = \frac{n_\varepsilon}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} R - \frac{(\beta_\varepsilon n_\varepsilon)^{1/2}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \le 0 \quad \text{in} \quad \mathbb{R}^d \times (0,\infty). \tag{24}$$

It follows from the maximum principle that

$$v_\varepsilon^A \le C_A + \varepsilon\ln(2) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty).$$

Next we show that $\underline{v}^A$ is a supersolution of (22). Since $u_m < -A$ and

$$\frac{(\beta_\varepsilon n_\varepsilon)^{1/2}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \le \frac{(\beta_\varepsilon n_\varepsilon)^{1/2}}{2n_\varepsilon \exp(\frac{-A}{\varepsilon})^{1/2}} = \frac{1}{2}\exp(\frac{u_m + A}{2\varepsilon}),$$

as $\varepsilon \to 0$ and uniformly on $\mathbb{R}^d \times (0,\infty)$, we have

$$\frac{(\beta_\varepsilon n_\varepsilon)^{1/2}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \to 0. \tag{25}$$

From (16), (24) and

$$0 \le \frac{n_\varepsilon}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \le 1,$$

we then deduce that, in $\mathbb{R}^d \times (0,\infty)$,

$$v^A_{\varepsilon,t} - \varepsilon\Delta v^A_\varepsilon - |Dv^A_\varepsilon|^2 \ge R - O(\varepsilon), \tag{26}$$

while by the definition of v^A_ε we also have

$$v^A_\varepsilon + A \ge 0. \tag{27}$$

Combining (26) and (27) and using the basic stability properties of the viscosity solutions (see [2]) we find that the lower semicontinuous function $\underline{v}^A$ is a viscosity supersolution of (22).

To prove that $\overline{v}^A$ is a subsolution to (22), following classical arguments from the theory of viscosity solutions (see [2]), we fix a smooth ϕ and assume that $\overline{v}^A - \phi$ has a strict local maximum at (x_0,t_0). It follows that there exists a family, which for notational simplicity we denote again by ε, of points $(x_\varepsilon,t_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^d \times (0,\infty)$ such that $v^A_\varepsilon - \phi$ has a local maximum at $(x_\varepsilon,t_\varepsilon)$, and, as $\varepsilon \to 0$, $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ and $v^A_\varepsilon(x_\varepsilon,t_\varepsilon) \to \overline{v}^A(x_0,t_0)$.

We also know, still using (24) and (25), that v^A_ε solves

$$v^A_{\varepsilon,t} - \varepsilon\Delta v^A_\varepsilon - |Dv^A_\varepsilon|^2 = \big(1 - \exp(\frac{-A - v^A_\varepsilon}{\varepsilon})\big)R - O(\varepsilon).$$

It then follows that, at $(x_\varepsilon,t_\varepsilon)$,

$$\phi_t - \varepsilon\Delta\phi - |D\phi|^2 - \big(1 - \exp(\varepsilon^{-1}(-A - v^A_\varepsilon))R \le O(\varepsilon). \tag{28}$$

Recall that $\lim_{\varepsilon\to 0} v^A_\varepsilon(x_\varepsilon,t_\varepsilon) = \overline{v}^A(x_0,t_0) \ge -A$. Hence, if $\overline{v}^A(x_0,t_0) > -A$, then

$$\lim_{\varepsilon\to 0} \exp(\varepsilon^{-1}(-A - v^A_\varepsilon(x_\varepsilon,t_\varepsilon))) = 0.$$

From this and (28) we deduce that, if $\overline{v}^A(x_0,t_0) > -A$, then, at (x_0,t_0),

$$\phi_t - |D\phi|^2 - R \le 0.$$

Next we show that $\overline{v}$ and $\underline{v}$ satisfy the appropriate initial conditions. Indeed, in view of (6) and (23), we know that, as $\varepsilon \to 0$,

$$v^A_\varepsilon \to \max(-A, u^0) \quad \text{on} \quad \mathbb{R}^d \times \{0\}.$$

It also follows from a classical argument in theory of viscosity solutions ([2, 4]) that, on $\mathbb{R}^d \times \{0\}$,

$$\overline{v}^A - \max(-A, u^0) \leq 0 \quad \text{ and } \quad \underline{v}^A - \max(-A, u^0) \geq 0.$$

and, hence, $\overline{v}^A$ and $\underline{v}^A$ satisfy respectively the discontinuous viscosity subsolution and supersolution initial condition corresponding to (22).

We already know from the definition of $\overline{v}^A$ and $\underline{v}^A$ that $\underline{v}^A \leq \overline{v}^A$, while from the comparison property for (22) in the class of semicontinuous viscosity solutions (see [1, 2, 9]) we conclude from the steps above that $\overline{v}^A \leq \underline{v}^A$ in $\mathbb{R}^d \times (0, \infty)$. Hence $\underline{v}^A = \overline{v}^A = v^{A,1}$ is the unique continuous viscosity solution of (22) and, consequently, the families v_ε^A and $v_\varepsilon^{A,1}$ converge, as $\varepsilon \to 0$ and locally uniformly, to $v^{A,1}$.

Combining (3) and (21) we find

$$v_\varepsilon^{A,1} = u_\varepsilon^1 + \varepsilon \ln(1 + \exp^{\frac{-A - u_\varepsilon^1}{\varepsilon}}) \quad \text{ and } \quad v_\varepsilon^{A,1} = -A + \varepsilon \ln(1 + \exp^{\frac{A + u_\varepsilon^1}{\varepsilon}}).$$

Moreover, as we already explained it in the introduction (see (12)–(13)), we know that, as $\varepsilon \to 0$, $u_\varepsilon^1 \to u^1$ locally uniformly. Hence, always for $A < -u_m$, we obtain that, as $\varepsilon \to 0$,

$$v_\varepsilon^{A,1} \to \max(u^1, -A) \quad \text{ locally uniformly in } \quad \mathbb{R}^d \times [0, \infty).$$

It also follows that the family $(v_\varepsilon^A)_{\varepsilon>0}$ converges, as $\varepsilon \to 0$, locally uniformly to $v^{A,1} = \max(u^1, -A)$. $\square$

3 General rate

When R changes sign, the situation is much more complicated and (17) does not hold in general. In this case we are able to provide only inequalities for the half-relaxed limits of the family $(u_\varepsilon)_{\varepsilon>0}$. These estimates are used later to characterize the limit when R is positive.

Fix u_0, $\delta > 0$ and recall that u^1 is the solution of (13) with $u^1 = u_0$ on $\mathbb{R}^d \times \{0\}$. We introduce next the family $(u_i^\delta[u_0], \mathcal{C}_i^\delta[u_0], \Omega_i^\delta[u_0])_{i \in \mathbb{Z}^+}$ which is defined iteratively. To this end, for $i = 1$, let

$$u_1^\delta[u_0] = u^1, \ \mathcal{C}_1^\delta[u_0] = \mathbb{R}^d \times [0, \infty) \ \text{ and } \ \Omega_1^\delta[u_0] = \{(x,t) \in \mathbb{R}^d \times [0, \infty) : u_1^\delta[u_0](x,t) > u_m - \delta\}, \tag{29}$$

and, given $u_i^\delta[u_0], \mathcal{C}_i^\delta[u_0]$ and $\Omega_i^\delta[u_0]$, $u_{i+1}^\delta[u_0] : \mathbb{R}^d \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\begin{aligned} u_{i+1}^\delta[u_0](x,t) = \sup \quad & \left\{ \int_0^t \left[-\frac{|\dot{x}(s)|^2}{4} + R\left(x(s)\right) \right] ds + u_0\left(x(0)\right) \ : \right. \\ & \left. x \in \mathrm{C}^1([0,t]; \mathbb{R}^d), \ (x(s), s) \in \Omega_i^\delta[u_0] \text{ for all } s \in [0,t], \ x(t) = x \right\}, \end{aligned} \tag{30}$$

with

$$\mathcal{C}_{i+1}^\delta[u_0] = \{(x,t) \in \Omega_i^\delta[u_0] : \ u_{i+1}^\delta[u_0](x,t) > -\infty\} \tag{31}$$

and

$$\Omega_{i+1}^\delta[u_0] = \{(x,t) \in \Omega_i^\delta[u_0] : \ u_{i+1}^\delta[u_0](x,t) > u_m - \delta\} \subset \mathcal{C}_{i+1}^\delta[u_0]. \tag{32}$$

It follows that, in general, $\mathcal{C}_{i+1}^\delta[u_0] \subseteq \Omega_i^\delta[u_0]$. The inclusion may be, however, strict, i.e., they may exist points $(\bar{x}, \bar{t}) \in \Omega_i^\delta[u_0]$ which cannot be connected to $\mathbb{R}^d \times \{0\}$ by a C^1 trajectory staying, for all $s \in [0,t]$ in $\Omega_i^\delta[u^0]$. (See Figure 1.)

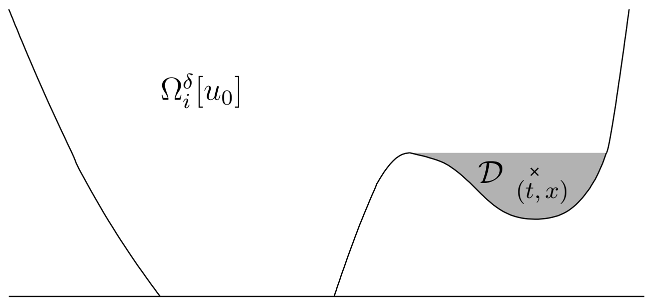

Figure 1: An example of the space-time set $\Omega_i^\delta[u_0]$. The point $(x,t) \in \Omega_i^\delta[u^0]$ cannot be connected to $\mathbb{R}^d \times \{0\}$ by a C^1 trajectory $(x(s),s)_{s\in[0,t]}$ staying within $\Omega_i^\delta[u^0]$. More generally, for the points in the grey area, called $\mathcal{D}$, there is no admissible trajectory. We have indeed $\mathcal{C}_{i+1}^\delta[u_0] = \Omega_i^\delta[u_0] \setminus \mathcal{D}$.

Moreover (5), (29) and classical considerations from the optimal control theory ([15, 13, 1, 8]) yield that, for all $i \in \mathbb{Z}^+$, the sets $\mathcal{C}_i^\delta[u_0]$ and Ω_i^δ are open and $u_i^\delta[u_0] \in \mathrm{C}(\mathcal{C}_i^\delta[u_0])$.

Note that the state constraint boundary condition, i.e., the requirement that the trajectories stay inside the domain, is hidden in the control formula. We do not write it, however, explicitly, because, to the best of our knowledge, there is no general theory, as in [18], for state constraint problem with time varying and nonsmooth domains. Note that in our context we have no regularity properties for these domains.

Given $\mathcal{C}_{i+1}^\delta[u_0]$ as in (31), it turns out that $u_{i+1}^\delta[u_0]$ is the minimal viscosity solution to

$$\begin{cases} u_{i+1,t}^\delta[u_0] = |Du_{i+1}^\delta[u_0]|^2 + R \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0], \\ u_{i+1}^\delta[u_0] = u_0 \quad \text{in } \mathcal{C}_{i+1}^\delta[u_0] \cap (\mathbb{R}^d \times \{0\}). \end{cases} \tag{33}$$

Indeed using standard arguments from optimal control theory (see, for example, [1, 2]), we may easily see that $u_{i+1}^\delta[u_0]$ satisfies the dynamic programming principle. The latter, as usual, implies that $u_{i+1}^\delta[u_0]$ is a viscosity solution of (33). The proof of the fact that $u_{i+1}^\delta[u_0]$ is a minimal solution to (33) in Appendix B.

The family $(u_i^\delta[u_0])_{i\in\mathbb{Z}^+,\delta>0}$ is nonincreasing in both i and δ. Therefore there exists $U^\delta[u_0] \geq -\infty$, which is itself nonincreasing in δ, such that, as $i \to +\infty$, $u_i^\delta[u_0] \searrow U^\delta[u_0] \quad \text{in} \quad \mathbb{R}^d \times [0,\infty)$.

Let $U[u_0]$ be the limit, as $\delta \to 0$, of the family $(U^\delta[u_0])_{\delta>0}$ and, for $\mu > 0$, consider the nonincreasing (in δ) family of sets

$$\Omega^\delta[u_0] = \bigcap_{i\in\mathbb{Z}^+} \Omega_i^\delta[u_0] \quad \text{and} \quad \Omega[u_0-\mu] = \bigcap_{\delta>0} \Omega^\delta[u_0-\mu]. \tag{34}$$

We have:

Theorem 3.1. *Let n_ε be the solution to (1), $u_\varepsilon = \varepsilon \ln(n_\varepsilon)$ and assume (5). Then, for any $\mu > 0$,*

$$\overline{u} \leq U[u_0] \quad \textit{in} \quad \mathbb{R}^d \times [0,\infty) \quad \textit{and} \quad U[u_0-\mu] + \mu \leq \underline{u} \quad \textit{in} \quad \Omega[u_0-\mu]. \tag{35}$$

Before we present the proof we remark that, by definition, $u_i^\delta[u_0] = -\infty$ in $\left(\mathcal{C}_i^\delta[u_0]\right)^c$. Therefore $U^\delta[u_0] = -\infty$ in $(\Omega^\delta[u_0])^c$ and, finally, $U[u_0] = -\infty$ in $(\Omega[u_0])^c = \left(\bigcap_{i,\delta} \mathcal{C}_i^\delta[u_0]\right)^c = \left(\bigcap_\delta \Omega^\delta[u_0]\right)^c$, and, hence,

$$\overline{u} = -\infty \quad \text{in} \quad (\Omega[u_0])^c.$$

Moreover, since $u_i^\delta[\cdot] \geq u_m - \delta$ in $\Omega^\delta[\cdot]$, by passing to the limit $i \to \infty$ and $\delta \to 0$ we also obtain

$$U[\cdot] \geq u_m \quad \text{in} \quad \Omega[\cdot].$$

An important question is whether, as $\mu \to 0$, $U[u_0 - \mu] \to U[u_0]$. This is, in general, not true. A counterexample can be found for $u^0 = u_m$ and $R > 0$. Then $\Omega_1^\delta[u_0 - \mu]$ cannot touch $\mathbb{R}^d \times \{0\}$ and $u_i^\delta[u_0 - \mu] \equiv -\infty$. Therefore $U[u_0 - \mu] \equiv -\infty$ for any $\mu > 0$. On the other hand, $u_i^\delta[u_0] > u_m$ and $U[u_0] = u^1$.

We continue with the

Proof of Theorem 3.1. First we show by induction that, for all $\delta > 0$ and $i \in \mathbb{Z}^+$, $\overline{u} \leq u_i^\delta[u_0]$.

Since n_ε^1 is a supersolution to (1), it follows from the comparison principle that $n_\varepsilon \leq n_\varepsilon^1$ and, hence, $\overline{u} \leq u_1^\delta[u_0] = u^1$.

Next we assume that $\overline{u} \leq u_i^\delta[u_0]$, and, arguing by contradiction, we show, following an argument similar to that in Section 2, that $\overline{u} \leq u_{i+1}^\delta[u_0] = -\infty$ in $(\Omega_i^\delta[u_0])^c$.

To this end, suppose that, for some $(x_0, t_0) \in (\Omega_i^\delta[u_0])^c$, $\overline{u}(x_0, t_0) > -\infty$. Since $\overline{u}$ is upper semi-continuous, there exists a family $(\phi_\alpha)_{\alpha>0}$ of smooth functions such that $\overline{u} - \phi_\alpha$ attains a strict local maximum at (x_α, t_α) and, as $\alpha \to 0$, $(x_\alpha, t_\alpha) \to (x_0, t_0)$, $\overline{u}(x_\alpha, t_\alpha) \geq \overline{u}(x_0, t_0)$, and, consequently, $\overline{u}(x_\alpha, t_\alpha) \to u(x_0, t_0)$. It follows that there exist points $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \in (\mathbb{R}^d \times (0, \infty))$ where $u_\varepsilon - \phi_\alpha$ attains a local maximum and, as $\varepsilon \to 0$, $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \to (x_\alpha, t_\alpha)$.

Moreover, in view of (4), at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$,

$$\phi_{\alpha,t} - \varepsilon \Delta \phi_\alpha - |D\phi_\alpha|^2 - R \leq -\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon)).$$

Letting $\varepsilon \to 0$ yields, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \leq \limsup_{\varepsilon \to 0} (-\exp[(2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}))]).$$

Since, by the definition of $\overline{u}$, we have $\limsup_{\varepsilon \to 0} u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \leq \overline{u}(x_\alpha, t_\alpha)$, the induction hypothesis yields that, for α small enough, $\overline{u}(x_\alpha, t_\alpha) \leq u_i^\delta[u_0](x_\alpha, t_\alpha) \leq u_m - \delta/2$.

It follows that

$$\limsup_{\varepsilon \to 0} (-\exp[(2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}))]) = -\infty,$$

and, hence, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \leq -\infty,$$

which, of course, is not possible because ϕ_α is a smooth function.

Hence we have $\overline{u} = -\infty$ in $(\Omega_i^\delta)^c$ and, in particular, $\overline{u} = -\infty$ on $\partial \Omega_i^\delta[u_0]$.

Next we show that

$$\overline{u} \leq u_{i+1}^\delta[u_0] = -\infty \quad \text{in} \quad (\mathcal{C}_{i+1}^\delta[u_0])^c.$$

To this end, let $(\bar{x}, \bar{t}) \in (\mathcal{C}_{i+1}^\delta[u_0])^c \setminus (\Omega_i^\delta[u_0])^c$. Note that the existence of such a point means that $(\bar{x}, \bar{t})$ cannot be connected to $\mathbb{R}^d \times \{0\}$ by a C^1-trajectory staying in $\Omega_i^\delta[u_0]$. Hence $(\bar{x}, \bar{t})$ belongs to a connected component $\mathcal{D}$ of $\omega_i^\delta[u_0] = \{(y, s) \in \Omega_i^\delta[u_0] : s \leq \bar{t}\}$, which does not touch $\mathbb{R}^d \times \{0\}$. (See Figure 1.)

Therefore $\partial_p \mathcal{D} \subset \partial \Omega_i^\delta[u_0]$, where $\partial_p \mathcal{D} = \{(y, s) \in \partial \mathcal{D} : s < \bar{t}\}$ is the parabolic boundary of $\mathcal{D}$. From the previous argument we obtain

$$\overline{u} = -\infty \qquad \text{on } \partial_p \mathcal{D}. \tag{36}$$

As in (21), for $A > 0$, we define w_ε^A by $n_\varepsilon + \exp\left(\frac{-A}{\varepsilon}\right) = \exp\left(\frac{w_\varepsilon^A}{\varepsilon}\right)$. Arguing as in the previous section, we deduce that, for all $A > 0$,

$$\overline{w}^A = \max(-A, \overline{u}) \quad \text{and} \quad \min(\overline{w}^A + A,\, \overline{w}_t^A - |D\overline{w}^A|^2 - R) \leq 0,$$

and, in view of (36),

$$\begin{cases} \min(\overline{w}^A + A,\, \overline{w}_t^A - |D\overline{w}^A|^2 - R) \leq 0 & \text{in} \quad \mathcal{D}, \\ \overline{w}^A = -A \quad \text{in} \quad \partial_p \mathcal{D}, \end{cases}$$

which admits $-A + C_1 t$ as a supersolution for some $C_1 > 0$.

It follows from the comparison principle that, for all $A > 0$,

$$\overline{u} \leq \overline{w}^A \leq -A + C_1 t \quad \text{in} \quad \mathcal{D}.$$

Letting $A \to \infty$ yields $\overline{u} = -\infty$ in $\mathcal{D}$ and, consequently, $\overline{u}(\bar{x}, \bar{t}) = -\infty$. Observe that $\overline{u} = -\infty$ in $(\mathcal{C}_{i+1}^\delta[u_0])^c$ implies that $\overline{u} = -\infty$ on $\partial\mathcal{C}_{i+1}^\delta[u_0] \cap (\mathbb{R}^d \times [0, \infty))$.

Finally we show that

$$\overline{u} \leq u_{i+1}^\delta[u_0] \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0].$$

To this end, define z_ε by $n_\varepsilon + \exp\left(\frac{u_{i+1}^\delta[u_0]}{\varepsilon}\right) = \exp\left(\frac{z_\varepsilon}{\varepsilon}\right)$ and notice that

$$z_\varepsilon = u_{i+1}^\delta[u_0] + \varepsilon \ln\left(\exp\left(\frac{u_\varepsilon - u_{i+1}^\delta[u_0]}{\varepsilon}\right) + 1\right) = u_\varepsilon + \varepsilon \ln\left(\exp\left(\frac{u_{i+1}^\delta[u_0] - u_\varepsilon}{\varepsilon}\right) + 1\right).$$

It follows that

$$\overline{z} = \max(\overline{u},\, u_{i+1}^\delta[u_0]).$$

We claim that $\overline{z}$ is a subsolution of

$$\overline{z}_t - |D\overline{z}|^2 - R \leq 0 \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0]. \tag{37}$$

Indeed $u_{i+1}^\delta[u_0]$ is a subsolution to (37) in $\mathcal{C}_{i+1}^\delta[u_0]$ by definition. Moreover if, for some $(\bar{x}, \bar{t}) \in \mathcal{C}_{i+1}^\delta[u_0]$, $\overline{u}(\bar{x}, \bar{t}) \neq -\infty$, using (4) and the stability of viscosity subsolutions, we find that $\overline{u}$ satisfies the viscosity subsolution criteria for (37) at $(\overline{x}, \overline{t})$. Finally, since the maximum of two subsolutions is always a subsolution, we obtain that $\overline{z}$ is a subsolution of (37).

We proceed by noticing that, since, in view of the above,

$$\overline{u} = -\infty \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta \cap (\mathbb{R}^d \times (0, +\infty)),$$

it follows that

$$\overline{z} = u_{i+1}^\delta[u_0] \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta[u_0]$$

and, hence,

$$\overline{z} \leq u_{i+1}^\delta[u_0] \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta[u_0].$$

Therefore, using again the comparison principle for (33), we obtain

$$\overline{z} \leq u_{i+1}^\delta[u_0] \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0]$$

and we conclude that $\overline{u} \leq u^\delta_{i+1}[u_0]$.

Finally, since for all $\delta > 0$ and $i \in \mathbb{Z}^+$, we have $\overline{u} \leq u^\delta_i[u_0]$ it follows that, for all $\delta > 0$, $\overline{u} \leq \lim_{i\to\infty} u^\delta_i[u_0] = U^\delta[u_0]$. After letting $\delta \to 0$ we obtain

$$\overline{u} \leq \lim_{\delta\to 0} U^\delta[u_0] = U[u_0] \quad \text{in} \quad \mathbb{R}^d \times (0,\infty),$$

which concludes the proof of the first part of the claim.

For the second part we need the following lemma which is essentially a result from [7] that we adapt to our context (see also [3]). Its proof is postponed to the end of this section.

Lemma 3.2. *For all $i \in \mathbb{Z}^+$ the lower semicontinuous function $v^\delta_i = \max(u^\delta_i[u_0-\mu] + 2\delta, \underline{u})$, is a supersolution of*

$$\begin{cases} v^\delta_{i,t} - |Dv^\delta_i|^2 - R \geq 0 \quad \textit{in} \quad \Omega^\delta_i[u_0-\mu], \\ v^\delta_i = u_0 \quad\quad \textit{in}\ \{u_0 - \mu > u_m - \delta\} \cap (\mathbb{R}^d \times \{0\}). \end{cases} \tag{38}$$

Since $u^\delta_{i+1}[u_0-\mu]$ is a minimal solution of (33) in $\mathcal{C}^\delta_{i+1}[u_0-\mu] \subset \Omega^\delta_i[u_0-\mu]$ with $u^\delta_{i+1}[u_0-\mu] = u_0 - \mu$ on $\mathbb{R}^d \times \{0\}$ (see Appendix B), it follows that

$$u^\delta_{i+1}[u_0-\mu] \leq v^\delta_i - \mu \quad \text{in} \quad \mathcal{C}^\delta_{i+1}[u_0-\mu],$$

and, hence,

$$u^\delta_{i+1}[u_0-\mu] + \mu \leq \max(u^\delta_i[u_0-\mu] + 2\delta, \underline{u}) \quad \text{in} \quad \mathcal{C}^\delta_{i+1}[u_0-\mu].$$

Letting $i \to \infty$ yields

$$U^\delta[u_0-\mu] + \mu \leq \max(U^\delta[u_0-\mu] + 2\delta, \underline{u}) \quad \text{in} \quad \Omega^\delta[u_0-\mu].$$

Choosing $\mu > 2\delta$ we also get

$$U^\delta[u_0-\mu] + 2\delta < U^\delta[u_0-\mu] + \mu \quad \text{in} \quad \Omega^\delta[u_0-\mu],$$

and, therefore,

$$U^\delta[u_0-\mu] + \mu \leq \underline{u} \quad \text{in} \quad \Omega^\delta[u_0-\mu].$$

Finally letting $\delta \to 0$ we obtain

$$U[u_0-\mu] + \mu = \lim_{\delta\to 0} U^\delta[u_0-\mu] + \mu \leq \underline{u} \quad \text{in} \quad \Omega[u_0-\mu].$$

□

We conclude with the

Proof of Lemma 3.2. The key idea of the proof comes from [7] and [3] and relies on the property that, for concave Hamiltonians, the maximum of two supersolutions is supersolution. Here we reprove this fact in the context of semicontinuous supersolutions in a space-time domain.

To this end, fix $i \in \mathbb{Z}^+$ and $(x,t) \in \mathcal{C}^\delta_i[u_0-\mu]$. Since $\mathcal{C}^\delta_i[u_0-\mu]$ is an open set, there exists $\rho > 0$ such that $B_\rho(x,t) \in \mathcal{C}^\delta_i[u_0-\mu]$, where $B_\rho(x,t)$ denotes the open ball of radius ρ centered at (x,t).

For $\alpha > 0$, we define

$$u_i^{\delta,\alpha}(x,t) = \inf_{(y,s)\in B_\rho(x,t)} \{u_i^\delta[u_0-\mu](y,s) + (2\alpha)^{-1}(|x-y|^2 + |t-s|^2)\} \quad \text{and} \quad u_i^{\delta,\alpha,\beta} = u_i^{\delta,\alpha} * \chi_\beta,$$

where χ_β is a standard smoothing mollifier.

Since $u_i^{\delta,\alpha}$ is an inf-convolution of the continuous function u_i^δ (see [2]), it is locally Lipschitz continuous and semi-concave with semi-concavity constant $1/\alpha$.

It follows that $u_i^{\delta,\alpha,\beta}$ is a smooth semi-concave function with semi-concavity constant $1/\alpha$ and

$$\liminf_{\substack{(y,s)\to(\bar{y},\bar{s})\\ \alpha,\,\beta\to 0}} u_i^{\delta,\alpha,\beta}(y,s) = u_i^\delta[u_0-\mu](\bar{y},\bar{s}).$$

Finally, using Jensen's inequality and the concavity of the Hamiltonian, we obtain that, for some $K>0$, $u_i^{\delta,\alpha,\beta}$ is a smooth and, hence, a classical supersolution to

$$u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2 - R*\chi_\beta \geq -K\alpha - \varepsilon/\alpha \quad \text{in} \quad B_\rho(x,t). \tag{39}$$

To prove (38) we show that the smooth approximations $v_i^{\delta,\alpha,\beta,\varepsilon}$ of v_i^δ in $B_\rho(x,t)$ given by

$$n_\varepsilon + \exp\Big(\frac{u_i^{\delta,\alpha,\beta}+2\delta}{\varepsilon}\Big) = \exp\Big(\frac{v_i^{\delta,\alpha,\beta,\varepsilon}}{\varepsilon}\Big) \tag{40}$$

are almost supersolutions to (38) for α, β and ε small. Notice that in (40) we use 2δ instead of δ.

Replacing n_ε by $\exp\big(\frac{v_i^{\delta,\alpha,\beta,\varepsilon}}{\varepsilon}\big) - \exp\big(\frac{u_i^{\delta,\alpha,\beta}+2\delta}{\varepsilon}\big)$ in (1) we get

$$\begin{aligned} Rn_\varepsilon - \beta_\varepsilon\sqrt{n_\varepsilon} = &\big(v_{i,t}^{\delta,\alpha,\beta,\varepsilon} - \varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2\big)\exp(\varepsilon^{-1}v_i^{\delta,\alpha,\beta}) \\ &- \big(u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2\big)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta)), \end{aligned}$$

and, in view of (40),

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta,\varepsilon} &- \varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2 \\ &= (u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2 - R*\chi_\beta)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta - v_i^{\delta,\alpha,\beta,\varepsilon})) \\ &+ (R*\chi_\beta - R)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta - v_i^{\delta,\alpha,\beta})) + R - \beta_\varepsilon n_\varepsilon^{1/2}\exp(-\varepsilon^{-1}v_i^{\delta,\alpha,\beta,\varepsilon}). \end{aligned}$$

Using that, in view of (40), $\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta-v_i^{\delta,\alpha,\beta,\varepsilon}))\leq 1$, and (39) we find

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta,\varepsilon} - &\varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2 - R \geq -K\alpha - \varepsilon/\alpha \\ &+ (R*\chi_\beta - R)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta - v_i^{\delta,\alpha,\beta})) - \beta_\varepsilon n_\varepsilon^{1/2}\exp(\varepsilon^{-1}v_i^{\delta,\alpha,\beta}). \end{aligned}$$

Define

$$v_i^{\delta,\alpha,\beta}(\bar{y},\bar{s}) = \liminf_{\substack{\varepsilon\to 0\\ (y,s)\to(\bar{y},\bar{s})}} v_i^{\delta,\alpha,\beta,\varepsilon}(y,s).$$

Letting $\varepsilon\to 0$ and using the stability of viscosity supersolutions we obtain

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta} - |Dv_i^{\delta,\alpha,\beta}|^2 - R \geq -K\alpha& \\ + \liminf_{\substack{\varepsilon\to 0\\ (y,s)\to(\bar{y},\bar{s})}} &[(R*\chi_\beta - R)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta}+2\delta - v_i^{\delta,\alpha,\beta})) - \beta_\varepsilon n_\varepsilon^{1/2}\exp(\varepsilon^{-1}v_i^{\delta,\alpha,\beta})]. \end{aligned} \tag{41}$$

Recalling that $u_i^\delta + 2\delta > u_m + \delta$ in $\Omega_i^\delta[u_0 - \mu]$, we deduce that, as $\varepsilon, \alpha, \beta \to 0$, in $B_\rho(x,t)$,

$$\beta_\varepsilon n_\varepsilon^{1/2} \exp(-\varepsilon^{-1} v_i^{\delta,\varepsilon}) = \beta_\varepsilon n_\varepsilon^{1/2} (n_\varepsilon + \exp(\varepsilon^{-1} u_i^{\delta,\varepsilon}))^{-1} \leq (1/2)\beta_\varepsilon \exp(-(2\varepsilon)^{-1} u_i^{\delta,\alpha,\beta}) \to 0. \tag{42}$$

Moreover, as $\varepsilon, \beta \to 0$, we also have

$$R * \chi_\beta - R \to 0, \quad \exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta})) < 1 \quad \text{and } v_i^\delta(\bar{\eta}, \bar{s}) = \liminf_{\substack{\alpha,\beta\to 0 \\ (y,s)\to(\bar{y},\bar{s})}} v_i^{\delta,\alpha,\beta}(y,s). \tag{43}$$

Using (41), (42), (43), and the stability of viscosity supersolutions we find

$$v_{i,t}^\delta - |Dv_i^\delta|^2 - R \geq 0 \quad \text{in} \quad B_\rho(x,t).$$

Since all the above hold for all $(x,t) \in \Omega_i^\delta[u_0 - \mu]$, it follows that the lower semicontinuous function v_i^δ is a supersolution to

$$v_{i,t}^\delta - |Dv_i^\delta|^2 - R \geq 0 \quad \text{in} \quad \Omega_i^\delta[u_0 - \mu], \;\; v_i^\delta = u_0 \quad \text{for} \quad \{u_0 - \mu > u_m - \delta\} \cap (\mathbb{R}^d \times \{0\}).$$

□

4 Constant rate

Here we assume that the rate is a constant, i.e.,

$$R(x) = R \quad \text{in } \mathbb{R}^d, \tag{44}$$

and, in addition, setting $O = \{x \in \mathbb{R}^d : \, u_0(x) > u_m\}$, we have

$$\overline{O} = \{x \in \mathbb{R}^d : \, u_0(x) \geq u_m\}. \tag{45}$$

We have:

Theorem 4.1. *Assume (44) and (45). Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = U[u_0](x,t) \quad \textit{locally uniformly in } \Big(\mathbb{R}^d \times [0,\infty)\Big) \setminus \{(x,t) \,|\, U[u_0](x,t) = u_m\}, \tag{46}$$

with

$$\Omega[u_0] = \{(x,t) | \sup_{y\in\overline{O}} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} \geq u_m\}, \tag{47}$$

and

$$U[u_0] = \begin{cases} \sup_{y\in\overline{O}} \left\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\right\} & \textit{if} \quad (x,t) \in \Omega[u_0], \\ -\infty \quad \textit{otherwise.} \end{cases} \tag{48}$$

We notice that, if $R < 0$, then one can obtain (46) from (17) and the dynamic programming principle. We also remark that, in particular, Theorem 4.1 shows that the limit of the family $(u_\varepsilon)_{\varepsilon>0}$ is not, in general, given by (17). We refer to Appendix A for an explicit example.

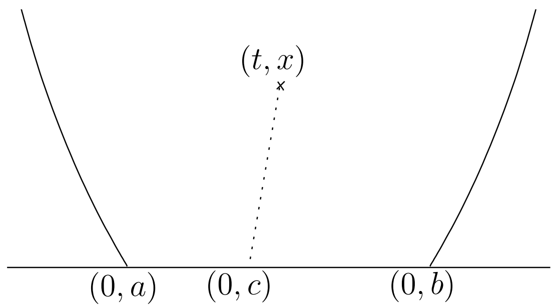

Figure 2: The case with $R(x) = R$ a positive constant, $\{x \in \mathbb{R} | \, u_0(x) > u_m\} = (a,b)$ and $u_0(\cdot) \geq u_m$ on an interval $[a,b]$. Then $\Omega = \cup_{d\in[a,b]}\{(x,t) \,|\, -\frac{|x-d|^2}{4t} + Rt + u_0(d) \geq u_m\}$, the optimal trajectories are straight lines and $U(x,t) = -\frac{|x-c|^2}{4t} + Rt + u_0(c)$, where c is a point where the maximum in (48) is attained.

Proof of Theorem 4.1. When the rate R is constant, after one iteration of (30), (31) and (32) we find both the set $\Omega^\delta[u_0]$ and the function $U^\delta[u_0]$, since for all $i > 1$, $j > 2$ and $\delta > 0$, $\Omega_i^\delta[u_0] = \mathcal{C}_j^\delta[u_0] = \Omega^\delta[u_0]$ and $u_i^\delta[u_0] = U^\delta[u_0]$.

Indeed, every optimal trajectory in $\mathcal{C}_2^\delta$ is a straight line connecting a point in Ω_2^δ to a point in $I^\delta = \{x \in \mathbb{R}^d : \, u_0(x) > u_m - \delta\}$ and, hence, it is included in Ω_2^δ. This follows from the observation that

$$\phi(x,t) = -\frac{|x-c|^2}{4t} + Rt + u_0(c)$$

is concave in (x,t) and, therefore, all the optimal trajectories of the points in Ω_2^δ are included in Ω_2^δ. It follows that $\Omega_2^\delta = \mathcal{C}_3^\delta$, $u_2^\delta = u_3^\delta$ and consequently $\Omega_2^\delta = \Omega_3^\delta$. By iteration we obtain, for all $i > 2$, $\Omega_2^\delta = \Omega_i^\delta = \Omega^\delta$ and $u_2^\delta = u_i^\delta = U^\delta$.

Using (32) and (30) we see that, for all $i \geq 2$,

$$\Omega^\delta[u_0] = \Omega_i^\delta[u_0] = \{(x,t) : \sup_{y\in I^\delta} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} > u_m - \delta\}, \tag{49}$$

and

$$U^\delta[u_0] = u_i^\delta[u_0](x,t) = \begin{cases} \sup_{y\in I^\delta} \left\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\right\} & \text{if} \quad (x,t) \in \Omega^\delta[u_0], \\ -\infty \quad \text{otherwise.} \end{cases} \tag{50}$$

It is easy to verify that (33) holds, since, for all $i > 2$ and $\delta > 0$,

$$u_{i,t}^\delta[u_0] - |Du_i^\delta[u_0]|^2 - R = 0 \quad \text{in} \quad \Omega_i^\delta[u_0] = \mathcal{C}_{i+1}^\delta[u_0].$$

Letting $\delta \to 0$ in (49) and (50) we obtain (47)–(48). (See Figure 2.)

We also have

$$\Omega[u_0 - \mu] = \bigcap_{\delta>0} \Omega^\delta[u_0 - \mu] = \{(x,t) : \sup_{y\in J^\mu} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} \geq u_m + \mu\},$$

with

$$J^\mu = \{(x,t) : \, u_0 \geq u_m + \mu\}.$$

It follows that

$$\cup_{\mu>0}\,\Omega[u_0-\mu] = \{(x,t) : \sup_{y\in O}\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} > u_m\}, \tag{51}$$

and

$$\lim_{\mu\to 0^+} U[u_0-\mu] = \begin{cases} \sup_{y\in O}\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} \quad \text{for} \quad (x,t)\in \cup_{\mu>0}\Omega[u_0-\mu], \\ \\ -\infty \quad \text{otherwise.} \end{cases} \tag{52}$$

We also notice that

$$\sup_{y\in O}\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} = \sup_{y\in \bar{O}}\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\}. \tag{53}$$

Comparing (47), (48) with (51), (52) and using (53) we deduce that

$$\lim_{\mu\to 0} U[u_0-\mu](x,t) = U[u_0](x,t) \quad \text{for} \quad U[u_0](x,t) \neq u_m,$$

and, consequently,

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = U[u_0](x,t) \quad \text{locally uniformly in } \Big(\mathbb{R}^d\times[0,\infty)\Big)\setminus\{(x,t)\,|\,U[u_0](x,t)=u_m\}\,.$$

□

5 Strictly positive rate

In this section we study the limiting behavior of the family $(u_\varepsilon)_{\varepsilon>0}$ when

$$R \geq a > 0 \quad \text{in} \quad \mathbb{R}^d, \tag{54}$$

and show that, in general, the limit is not given by (17).

For this we need to assume that, for sufficiently small $\mu > \delta > 0$, there exists $\rho_{\delta,\mu} > 0$ such that

$$\lim_{\mu\to 0}\lim_{\delta\to 0}\rho_{\delta,\mu} = 0 \quad \text{and, if} \quad u_0(y) > u_m - \delta, \quad \text{then} \quad \sup_{|y-z|\leq\rho_{\delta,\mu}} u_0(z) > u_m - \delta + \mu. \tag{55}$$

Notice that it is important that $\rho_{\delta,\mu}$ is chosen independently of y. If $u_0 \in \mathrm{C}^1$, (55) implies u_m is never a local maximum of u.

We have

Theorem 5.1. *Assume (54) and (55). Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = U[u_0] \quad \textit{locally uniformly in} \quad \cup_{\mu>0}\,\Omega[u_0-\mu]. \tag{56}$$

Recall that, in view of Theorem 3.1, we already know that $\lim_{\varepsilon\to 0} u_\varepsilon = -\infty$ in $\Omega[u_0]^c$.

Proof of Theorem 5.1. For $h > \bar{h} = \frac{\mu}{2a} + \frac{1}{2}\sqrt{\frac{\mu^2}{a^2} + \frac{\rho^2_{\delta,\mu}}{a}}$, $(x,t) \in \mathbb{R}^d \times [0,\infty)$, $i \geq 1$ and $\mu, \delta > 0$, we have

$$u_i^\delta[u_0](x,t) \leq u_i^\delta[u_0 - \mu](x, t+h). \tag{57}$$

We postpone the proof of this inequality to Appendix C and we continue with the ongoing one.

Letting $i \to +\infty$ and $\delta, \mu \to 0$ we find, for all $h > 0$ and $t > 0$,

$$U[u_0](\cdot,\cdot) \leq \lim_{\mu \to 0^+} U[u_0 - \mu](\cdot, \cdot + h). \tag{58}$$

Hence, for all $(x,t) \in \cup_{\mu>0}\Omega[u_0 - \mu]$,

$$U[u_0](x,t) \leq \lim_{\mu \to 0^+} U[u_0 - \mu](x, t+h) \leq \underline{u}(x, t+h) \leq \overline{u}(x, t+h),$$

$$U[u_0](x,t) \leq \liminf_{h \to 0^+} \underline{u}(x, t+h) \leq \limsup_{h \to 0^+} \overline{u}(x, t+h).$$

The definitions of $\underline{u}$ and $\overline{u}$ also imply that

$$\liminf_{h \to 0^+} \underline{u}(x, t+h) = \underline{u}(x,t) \quad \text{and} \quad \limsup_{h \to 0^+} \overline{u}(x, t+h) = \overline{u}(x,t).$$

Combining all the above we obtain

$$U[u_0] \leq \underline{u} \leq \overline{u} \quad \text{in} \quad \cup_{\mu>0}\, \Omega[u_0 - \mu].$$

This last inequality and (35) yield $\underline{u} = \overline{u} = U[u_0]$ in $\cup_{\mu>0}\, \Omega[u_0 - \mu]$, and, hence,

$$\lim_{\varepsilon \to 0} u_\varepsilon = U[u_0] \quad \text{in} \quad \cup_{\mu>0}\, \Omega[u_0 - \mu].$$

□

6 Conclusions

We showed that the local uniform limit, as $\varepsilon \to 0$, for the parabolic problem (1) with finite time extinction is naturally analyzed using the Hopf-Cole change of variables (3). The formal limit is the variant (7) of the standard eikonal equation. The new feature is the resulting quasi-variational inequality with an obstacle that depends on the solution itself.

The quasi-variational inequality admits many solutions (see Appendix A) and the difficulty is to select the correct additional information. This is easy when the rate R is negative, as shown in Section 2. Indeed, in this case it is enough to enforce the Dirichlet boundary condition on the boundary of the unknown open set Ω where the liminf of the family $(u_\varepsilon)_{\varepsilon>0}$ is finite. This is due to the fact that, for concave Hamiltonians, the supremum of two supersolutions is still a supersolution.

When the rate R is positive we do not have easy supersolutions at hand, and the answer is more elaborate. It requires an induction argument which allows us to identify again the limit of the family $(u_\varepsilon)_{\varepsilon>0}$. The key ingredient is a free boundary problem defined through the level set of the solution. The boundary condition for the resulting equation involves state constraints which leads us to study the problem using the related control problem.

If the growth/death rate R changes sign, we can only bound from above and below the half-relaxed limits of the family $(u_\varepsilon)_{\varepsilon>0}$ by sub and supersolutions $\bar{u}$ and $\underline{u}$ respectively.

In terms of the biological motivation, our results qualitatively mean that the specific form of the survival threshold (a square root here) is irrelevant for the asymptotic problem. It also shows that the solution is deeply influenced by the survival threshold except when R is nonpositive. This confirms earlier numerical simulations in [14].

We conjecture that these upper and lower solutions are in fact equal and the correct setting (implying uniqueness) is to find a pair (u, Ω) for which we can impose both Dirichlet and state constraints boundary conditions. Both establishing directly these boundary conditions for the half-limits of the family $(u_\varepsilon)_{\varepsilon>0}$ as well as developing a theory of state constraints boundary conditions for time varying, non-smooth domains are challenging mathematical issues.

A Non-uniqueness

To explain the difficulty associated with (7), we present here counter-examples for uniqueness and elaborate further conditions. Recall that the problem is to find pairs (u, Ω) such that u is a viscosity solutions to (7).

A first source for non-uniqueness is the value of u on $\partial\Omega$. Indeed assume that R and u_0 are such that there exists a unique viscosity solution u^1 of (13) or, more generally, with u^1 defined in (12) and (11). For all $\eta \geq u_m$, we introduce the pair (w_η, Ω_η) given by

$$\Omega_\eta = \{(x,t) : \; u^1(x,t) \geq \eta\} \quad \text{and} \quad w_\eta(x,t) = \begin{cases} u^1(x,t) & \text{if} \quad (x,t) \in \Omega_\eta, \\ -\infty & \text{otherwise.} \end{cases}$$

It can be easily verified that (w_η, Ω_η) is a viscosity solution of (7). In order to avoid this artefact, one can add the Dirchlet boundary condition (8) which appeared throughout our constructions. However in the next example we see that this Dirichlet condition is not enough to obtain uniqueness. In fact a state constraint boundary condition is hidden behind the property $u^1 = -\infty$ in the complement of $\overline{\Omega}_\eta$ and we do not take it into account here.

Let

$$R(x) = 1 \quad \text{and} \quad u_0(x) = -x^2.$$

A simple computation shows that the solution u^1 to (13) is given by

$$u^1(x,t) = t - \frac{x^2}{1+4t}.$$

Therefore the first truncation of u^1, given by

$$\widetilde{u}(x,t) = \begin{cases} t - \frac{x^2}{1+4t} & \text{for} \quad t - \frac{x^2}{1+4t} \geq u_m, \\ -\infty & \text{otherwise,} \end{cases}$$

with

$$\widetilde{\Omega} = \{(x,t) : \widetilde{u}(x,t) > -\infty\},$$

is a viscosity solution of (7). As a matter of fact this is the maximal subsolution to (7), (8) but it does not satisfy the state constraint boundary condition. To see this choose $u_m = -0.04$. The point $(1,2)$

is included in $\widetilde{\Omega}$ since $\widetilde{u}(1,2) = 0.2 > -0.04$. The optimal trajectory associated to this point, giving the value $\widetilde{u}(1,2) = 0.2$, is the straight line connecting $(0, 0.4)$ to $(1, 2)$. But $u_0(0.4) = -0.16 < -0.04$. So the point $(0, 0.4)$ is not included in $\widetilde{\Omega}$. Therefore a part of the optimal trajectory of the point $(1, 2)$ is not included in $\widetilde{\Omega}$. Hence $\widetilde{u}$ does not satisfy the state constraint condition.

Following the arguments in Section 4 we can find a viscosity solution to (7) and (8). Indeed using (48) it is possible to compute explicitly the function $U[u_0] = \lim_{\delta\to 0} U^\delta[u_0] = \lim_{\delta\to 0} u_2^\delta[u_0]$ to find

$$\breve{u}(x,t) = \begin{cases} t - \frac{x^2}{1+4t} \quad \text{if} \quad -\frac{x^2}{(1+4t)^2} \geq u_m,\ t - \frac{x^2}{1+4t} \geq u_m, \\ t - \frac{(x-\sqrt{-u_m})^2}{4t} + u_m \quad \text{if} \quad x > 0,\ -\frac{x^2}{(1+4t)^2} \leq u_m,\ t \geq \frac{(x-\sqrt{-u_m})^2}{4t}, \\ t - \frac{(x+\sqrt{-u_m})^2}{4t} + u_m \quad \text{if} \quad x < 0,\ -\frac{x^2}{(1+4t)^2} \leq u_m,\ t \geq \frac{(x+\sqrt{-u_m})^2}{4t}, \\ -\infty \quad \text{otherwise}, \end{cases}$$

with

$$\breve{\Omega} = \{(x,t) : \breve{u}(x,t) > -\infty\}.$$

From Theorem 4.1 we know that $\breve{u}$ is indeed the pointwise limit of the family $(u_\varepsilon)_{\varepsilon>0}$ outside the exceptional set $\{(x,t) : \breve{u}(x,t) = u_m\}$.

However, in general $\widetilde{u} \neq \breve{u}$. Consider, for instance, the value $u_m = -0.04$. Then

$$\widetilde{u}(2,1) = 0.2, \quad \breve{u}(2,1) = 0.15, \quad \widetilde{u}(2.21,1) = 0.02, \quad \breve{u}(2.21,1) = -\infty.$$

and, consequently, $\breve{\Omega} \subsetneq \widetilde{\Omega}$.

On the other hand, according to Section 4, the state constraint boundary condition is satisfied for $\breve{u}$, which motivates our conjecture in Section 6.

B $u_i^\delta[u_0]$ is a minimal solution of (33) in $\mathcal{C}_i^\delta[u_0]$

Here we prove that $u_i^\delta[u_0]$ is a minimal solution of (33) in $\mathcal{C}_i^\delta[u_0]$ by considering a supersolution $w \in \mathcal{C}_i^\delta[u_0]$ of (33) and showing that

$$u_i^\delta[u_0] \leq w \quad \text{in} \quad \mathcal{C}_i^\delta[u_0]. \tag{59}$$

To this end, we fix $(x,t) \in \mathcal{C}_i^\delta[u_0]$ and assume that $(\gamma(\cdot), \cdot) : [0,t] \to \Omega_{i-1}^\delta[u_0]$ is a C^1-trajectory with $(\gamma(t), t) = (x,t)$. Since $\mathcal{C}_i^\delta[u_0]$ is the set of points that can be connected by a C^1-trajectory in $\Omega_{i-1}^\delta[u_0]$ to some point in $\mathbb{R}^d \times \{0\}$, it follows that γ is included in $\mathcal{C}_i^\delta[u_0]$.

For the supersolution w, we define, for $s \in [0,t]$, the (clearly) lower semicontinuous function $\varphi(s) = w(\gamma(s), s)$ and we observe that φ is a viscosity supersolution of

$$\varphi' \geq -\frac{|\dot{\gamma}|^2}{4} + R(\gamma) \qquad \text{in} \quad (0,t). \tag{60}$$

We postpone the proof of this claim to the end of the present paragraph and we proceed noticing that the function

$$\psi(t) = \int_0^t \big(-\frac{|\dot{\gamma}(s)|^2}{4} + R(\gamma(s))\big) ds + w(\gamma(0), 0),$$

is a subsolution of (60). Then using the standard comparison principle of viscosity solutions we obtain

$$w(x,t)=\varphi(t)\geq \int_0^t \big(-\frac{|\dot\gamma(s)|^2}{4}+R(\gamma(s))\big)ds+u_0(\gamma(0)),$$

and, since this is true for any C^1-trajectory γ and any $(x,t)\in\mathcal{C}_i^\delta[u_0]$, (59) follows.

It remains to prove (60). Let $\phi\in\mathrm{C}^1((0,t))$ be a test function, assume that $\bar t$ is a strict minimum point of $\varphi-\phi$ and consider the function

$$F_\mu(y,t)=w(y,t)-\phi(t)+\frac{|y-\gamma(t)|^2}{\mu^2}+(t-\bar t)^2,$$

which attains a local minimum at a point (y_μ,t_μ) such that, as $\mu\to 0$,

$$t_\mu-\bar t\to 0 \quad \text{ and } \quad \frac{|y_\mu-\gamma(t_\mu)|^2}{\mu^2}\to 0. \tag{61}$$

Since w is a supersolution to (33), we have

$$\phi'(t_\mu)+\frac{2\big(\gamma(t_\mu)-y_\mu\big)}{\mu^2}\cdot\dot\gamma(t_\mu)+2(t_\mu-\bar t)\geq \Big|\frac{2(y_\mu-\gamma(t_\mu))}{\mu^2}\Big|^2+R(y_\mu).$$

It is immediate that

$$\phi'(t_\mu)+2(t_\mu-\bar t)\geq -\frac{|\dot\gamma(t_\mu)|^2}{4}+R(y_\mu),$$

and, after letting $\mu\to 0$, we conclude using (61).

C The proof of (57)

We prove by induction on i that, for all $h>\bar h=\frac{\mu}{2a}+\frac{1}{2}\sqrt{\frac{\mu^2}{a^2}+\frac{\rho^2_{\delta,\mu}}{a}}$, $i>1$, $\delta>0$, and $(x,t)\in\mathbb{R}^d\times[0,\infty)$,

$$u_i^\delta[u_0](x,t)\leq u_i^\delta[u_0-\mu](x,t+h).$$

Recall that $u_1^\delta[u_0]=u^1[u_0]$ and $u_1^\delta[u_0-\mu]=u^1[u_0-\mu]=u^1[u_0]-\mu$, where $u^1[u_0]$ is the solution of (13). Moreover (54) yields

$$u^1[u_0](\cdot,t)+ah-\mu\leq u^1[u_0](\cdot,t+h)-\mu=u^1[u_0-\mu](\cdot,t+h).$$

Therefore, for all $h>\bar h\geq\mu/a$, we have

$$u^1[u_0](\cdot,t)\leq u^1[u_0-\mu](\cdot,t+h),$$

and, consequently,

$$u_1^\delta[u_0](\cdot,t)\leq u_1^\delta[u_0-\mu](\cdot,t+h).$$

If, for all $h>\bar h$ and $t>0$,

$$u_i^\delta[u_0](\cdot,t)\leq u_i^\delta[u_0-\mu](\cdot,t+h),$$

it follows that, for all $h>\bar h$,

$$\Omega_i^\delta[u_0]+he_t\subset\Omega_i^\delta[u_0-\mu], \tag{62}$$

where e_t is the unit vector in the direction of time axis.

Fix $(x,t) \in \mathcal{C}^\delta_{i+1}[u_0] \subset \Omega^\delta_i[u_0]$ and let γ be a C^1-trajectory in $\Omega^\delta_i[u_0]$ connecting (x,t) to a point $(y,0)$ with $u_0(y) > u_m - \delta$. It follows from (55) that there exists $z \in \mathbb{R}^d$ such that $|z-y| < \rho_{\delta,\mu}$ and $u_0(z) > u_m - \delta + \mu$. Without loss of generality we can take $u_0(z) \geq u_0(y)$.

The claim is that the trajectory $\widetilde{\gamma} : [0, t+h] \to \mathbb{R}^d$ defined by

$$\widetilde{\gamma}(s) = \begin{cases} h^{-1}s(y-z) + z & \text{if} \quad 0 \leq s \leq h, \\ \gamma(s-h) & \text{for} \quad h < s \leq t+h, \end{cases} \tag{63}$$

is included in $\Omega^\delta_i[u_0 - \mu]$. Indeed notice that the choice of $\bar{h}$ yields, for all $h > \bar{h}$,

$$-\frac{|y-z|^2}{4h} + ah \geq \mu \geq 0.$$

Consequently, it follows from (54) and the choice of z that the straight line connecting (y,h) to $(z,0)$ is included in $\Omega^\delta[u_0 - \mu] = \cap_j \Omega^\delta_j[u_0 - \mu]$, and, in particular, in $\Omega^\delta_i[u_0 - \mu]$. Therefore, for all $0 \leq s \leq h$, the point $(\widetilde{\gamma}(s), s)$ is included in $\Omega^\delta_i[u_0 - \mu]$.

Moreover using (62) we find that $(\gamma(s), s+h) \in \Omega^\delta_i[u_0 - \mu]$ for all $s \geq 0$. Hence, for all $h < s$, $(\widetilde{\gamma}(s), s) \in \Omega^\delta_i[u_0 - \mu]$, and we conclude that $\widetilde{\gamma}$ is included in $\Omega^\delta_i[u_0 - \mu]$.

Next write

$$\begin{aligned} \int_0^{t+h} \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu = & \int_0^t \left(-\frac{|\dot{\gamma}(s)|^2}{4} + R(\gamma(s))\right)ds \\ & + \int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu. \end{aligned} \tag{64}$$

It follows that

$$\int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu \geq u_0(y). \tag{65}$$

If this is true, then using (30), (64) and (65) we obtain, for all $h > \bar{h}$ and $t > 0$,

$$u^\delta_{i+1}[u_0](\cdot, t) \leq u^\delta_{i+1}[u_0 - \mu](\cdot, t+h),$$

and we deduce (57).

It remains to prove (65). Since $R \geq a$, in view of (63), we have

$$u_0(z) \geq u_0(y), \quad \int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu \geq -\frac{|y-z|^2}{4h} + ah + u_0(z) - \mu,$$

and, for all $h > \bar{h}$,

$$-\frac{|y-z|^2}{4h} + ah \geq \mu.$$

References

[1] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[2] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1994.

[3] G. Barles. Discontinuous viscosity solutions of first order Hamilton-Jacobi Equations: A guided visit. *Nonlinear Analysis TMA*, Vol 20, N° 9, 1993, p 1123-1134.

[4] G. Barles, L. C. Evans, and P. E. Souganidis. Wavefront propagation for reaction-diffusion systems of PDE. *Duke Math. J.*, 61(3):835–858, 1990.

[5] G. Barles, S. Mirrahimi, and B. Perthame. Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result. *Methods Appl. Anal.*, 16(3):321–340, 2009.

[6] G. Barles and B. Perthame. Concentrations and constrained Hamilton-Jacobi equations arising in adaptive dynamics. *Contemp. Math.*, 439:57–68, 2007.

[7] E. N. Barron and R. Jensen. Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Comm. Partial Differential Equations*, 15(12):1713–1742, 1990.

[8] P. Cardaliaguet. A note on the regularity of solutions of Hamilton-Jacobi equations with superlinear growth in the gradient variable. *ESAIM: COCV*, 15(2):367–376, 2009.

[9] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[10] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, (1):1–42, 1983.

[11] L. C. Evans and B. K. Knerr. Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities. *Illinois J. Math.*, 23(1):153–166, 1979.

[12] L. C. Evans and P. E. Souganidis. A PDE approach to geometric optics for certain semilinear parabolic equations. *Indiana Univ. Math. J.*, 38(1):141–172, 1989.

[13] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[14] M. Gauduchon and B. Perthame. Survival thresholds and mortality rates in adaptive dynamics: conciliating deterministic and stochastic simulations. *Mathematical Medicine and Biology*, 27(3): 195–210, 2010.

[15] P. L. Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, Boston, 1982.

[16] B. Perthame and G. Barles. Dirac concentrations in Lotka-Volterra parabolic PDEs. *Indiana Univ. Math. J.*, 57(7):3275–3301, 2008.

[17] P. Quittner and P. Souplet. *Superlinear parabolic problems. Blow-up, global existence and steady state*. Birkhäuser Advanced texts, 2007.

[18] H. M. Soner. Optimal control with state-space constraint. *SIAM J. Control Optim.*, 24(3):552–561, 1986.

[19] P. E. Souganidis. Front propagation: theory and applications. In *Viscosity solutions and applications (Montecatini Terme, 1995)*, volume 1660 of *Lecture Notes in Math.*, pages 186–242. Springer, Berlin, 1997.